Covariance and Quantum Cosmology

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Abstract

In relativity, time is relative between reference frames. However, quantum mechanics requires a specific time coordinate in order to write an evolution equation for wave functions. This difference between the two theories leads to the problem of time in quantum gravity. One method to study quantum relativity is to interpret the dynamics of a matter field as a clock. In order to test the relationship between different reference frames, an isotropic cosmological model with two matter ingredients is introduced. One is given by a scalar field and one by vacuum energy or a cosmological constant. There are two matter fields, and thus two different Hamiltonians are derived from the respective clock rates. Semi-classical solutions are found for these equations and a comparison is made of the physical predictions that they imply.

Introduction

Quantum mechanics and general relativity are two of the most rigorously tested theories in modern physics. Quantum mechanics, which excels at describing the smallest constituents of the universe, has the power to explain many of the problems that baffled classical physicists on the small and every day scale. General relativity on the other hand, is best at describing the universe on a large scale. The framework of general relativity models gravitation on the geometric properties of a continuous space-time that spans the universe. Furthermore, general relativity is the basis of many modern cosmological models. Problems arise when one tries to incorporate quantum mechanics with theories of gravitation or cosmology, typically referred to as quantum gravity or quantum cosmology. The problems stem from the mathematical differences underlying quantum mechanics and general relativity. While general relativity requires space to be continuous, quantum mechanics requires that everything is “quantized", or to exist in small pieces. This fundamental difference may, on the surface, seem like something that can be dealt with easily. However, the problems that follow from this difference have challenged physicists for the past century.

This paper will deal with the theory of quantum cosmology. Quantum cosmology starts with a quantization of a structureless, homogeneous chunk of space as a first approximation to a “space-time" atom.

The Friedmann equations are a set of equations in physical cosmology that govern the expansion of space in homogeneous and isotropic models of the universe within the context of general relativity. Classically, the Friedmann equation is covariant so one is able to transform the time parameters without changing other physical parameters. Quantization, however, introduces correction terms to the classical model that makes it unclear whether or not we can still transform time in this way [1]. The motivation for this paper is the fact that while the time coordinate is not quantized since it is a coordinate and not directly observable, we are still able to model time using a method called deparameterization. Deparameterization involves using matter variables as an internal time. More specifically, we have one model in which we have two options as internal time. In this paper we try to solve them and subsequently compare their quantum evolutions. A paper by Styer [2] presents a method that can be expanded to find semi-classical solutions for these two cases. We apply this method and compare the physical predictions that they imply.

This paper is organized as follows. We start by setting up our isotropic cosmological model and defining the classical conditions of this model. We then consider quantization and utilize a Taylor expansion to find approximate solutions. Standard numerical analysis is used to find semi-classical solutions for our equations. We conclude by making comparisons between each matter clock's respective solution and discuss the physical implications of such comparisons.

Methods

Classical Model

If we assume coordinates x^a such that a=1,2,3, this implies that the coordinate volume $V_0=\int d^3x$ is dependent on our choice of coordinates. We can make the volume coordinate independent by using a metric tensor h_{ab} and defining $V=\int \sqrt{\det h_{ab}}\, d^3x$ which instead depends only on the metric. This is considered a physical field in general relativity.

The Friedmann equation for a homogeneous, isotropic, flat chunk of space, ($h_{ab}=a(t)^2\,\delta_{ab}$) in Cartesian coordinates is

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\rho$$

where a(t) is the scale factor and ρ is the energy density of matter. For our purposes, V is proportional to a^3. The Friedmann equation is a constraint in quantum cosmology rather than an evolution generator so we have to use a method called deparameterization to determine evolution. The idea of deparameterization is to take a special choice of matter field coupled to space-time so that its homogeneous value can formally play the role of time. The most common example is a free, massless, scalar field ϕ with momentum p_ϕ and energy density $\rho = p_\phi/2a^6$.

Then from the Friedmann equation, we get

$$p_\phi(a, p_a) = \sqrt{\frac{4\pi G}{3c^2}}|ap_a|, \quad (V_0 = 1 \text{ for now})$$

It follows that the equations of motion are

$$\frac{da}{d\phi} = \frac{\partial p_\phi}{\partial p_a} = \sqrt{\frac{4\pi G}{3c^2}}a, \quad \frac{dp_a}{d\phi} = \frac{\partial p_\phi}{\partial a} = -\sqrt{\frac{4\pi G}{3c^2}}p_a$$

We now introduce a cosmological constant Λ and have the energy density,

$$\rho = \frac{p_\phi^2}{2a^6} + \Lambda.$$

Substituting this into the first part of the Friedmann equation and solving for p_ϕ gives us

$$p_\phi = \sqrt{2}V\sqrt{\frac{3}{8\pi G}H^2 - \Lambda}$$

Quantization

Now we consider the case where the cosmological constant is an operator,

$$\frac{3}{8\pi G}\hat{H}^2 = \hat{\rho} = \frac{1}{2}\frac{\hat{p}_\phi^2}{\hat{V}^2} + \hat{\Lambda}$$

This is appropriate to do because all commutators are zero. We introduce

$$\hat{C} = -\frac{3}{8\pi G}\hat{H}^2 + \frac{1}{2}\frac{\hat{p}_\phi^2}{\hat{V}^2} + \hat{\Lambda}$$

which is analogous to the Hamiltonian and a dummy variable T such that

$$\hat{\Lambda} = \hat{p}_T = \frac{\hbar}{i}\frac{\partial}{\partial T} \implies \hat{C} = -\frac{3}{8\pi G}\hat{H}^2 + \frac{1}{2}\frac{\hat{p}_\phi^2}{\hat{V}^2} + \frac{\hbar}{i}\frac{\partial}{\partial T}$$

and the condition that

$$\hat{C}\psi = 0 \implies i\hbar\frac{\partial \psi}{\partial T} = \left(-\frac{3}{8\pi G}\hat{H}^2 + \frac{1}{2}\frac{\hat{p}_\phi^2}{\hat{V}^2}\right)\psi$$

Application of Taylor Expansion

We will apply semi-classical methods to the Hamiltonians p_T and p_ϕ:

$$p_T = \frac{3}{8\pi G}H^2 - \frac{1}{2}\frac{p_\phi^2}{V^2}, \quad (p_\phi \text{ const.})$$

$$p_\phi = \sqrt{2}V\sqrt{\frac{3}{8\pi G}H^2 - \Lambda}, \quad (\Lambda \text{ const})$$

These Hamiltonians are not quadratic, which means that the equations obtained following the procedure outlined in Styer's paper [2] cannot be solved exactly. However, we can use several approximations. As a first step, we can use a Taylor expansion to second order, and apply methods as used in [2] to the resulting quadratic expression. We will see differences between using T and ϕ at this level.

Using a Taylor expansion of p to second order, we get

$$\hat{p} = p(\hat{H}, \hat{V}) = p\Big(\langle \hat{H} \rangle + (\hat{H} - \langle \hat{H} \rangle), \langle \hat{V} \rangle + (\hat{V} - \langle \hat{V} \rangle)\Big)$$

$$= p(\langle \hat{H} \rangle, \langle \hat{V} \rangle) + \frac{\partial p(\langle \hat{H} \rangle, \langle \hat{V} \rangle)}{\partial \langle \hat{H} \rangle}(\hat{H} - \langle \hat{H} \rangle) + \frac{\partial p(\langle \hat{H} \rangle, \langle \hat{V} \rangle)}{\partial \langle \hat{V} \rangle}(\hat{V} - \langle \hat{V} \rangle) + \frac{1}{2}\Bigg(\frac{\partial^2 p(\langle \hat{H} \rangle, \langle \hat{V} \rangle)}{\partial \langle \hat{H} \rangle^2}(\hat{H} - \langle \hat{H} \rangle)^2$$

$$+\frac{\partial^2 p(\langle \hat{H} \rangle, \langle \hat{V} \rangle)}{\partial \langle \hat{H} \rangle \langle \hat{V} \rangle}\Big((\hat{H} - \langle \hat{H} \rangle)(\hat{V} - \langle \hat{V} \rangle) + (\hat{V} - \langle \hat{V} \rangle)(\hat{H} - \langle \hat{H} \rangle)\Big) + \frac{\partial^2 p(\langle \hat{H} \rangle, \langle \hat{V} \rangle)}{\partial \langle \hat{V} \rangle^2}(\hat{V} - \langle \hat{V} \rangle)^2\Bigg) + \dots$$

Now we have to determine the time derivatives

$$\frac{d\langle \hat{V} \rangle}{dt}, \frac{d\langle \hat{H} \rangle}{dt}, \frac{d\langle \hat{V}^2 \rangle}{dt}, \frac{d\langle \hat{H}^2 \rangle}{dt}, \frac{d\langle \hat{H}\hat{V} + \hat{V}\hat{H} \rangle}{dt}$$

First, we compute the order terms for all of the time derivatives.

$$\frac{d\langle \hat{V} \rangle}{dt} = \sum_{\alpha=1}^{7} \frac{\langle [\hat{V}, \hat{p}_\alpha] \rangle}{i\hbar} = \frac{\partial p}{\partial \langle \hat{H} \rangle}, \quad \frac{d\langle \hat{H} \rangle}{dt} = \sum_{\alpha=1}^{7} \frac{\langle [\hat{H}, \hat{p}_\alpha] \rangle}{i\hbar} = -\frac{\partial p}{\partial \langle \hat{V} \rangle}$$

$$\frac{d\langle \hat{V}^2 \rangle}{dt} = \sum_{\alpha=1}^{7} \frac{\langle [\hat{V}^2, \hat{p}_\alpha] \rangle}{i\hbar} = \frac{\partial p}{\partial \langle \hat{H} \rangle} 2\langle \hat{V} \rangle + 2\frac{\partial^2 p}{\partial \langle \hat{H} \rangle^2}(\langle \hat{V}\hat{H} + \hat{H}\hat{V} \rangle - 2\langle \hat{V} \rangle \langle \hat{H} \rangle) + 2\frac{\partial^2 p}{\partial \langle \hat{H} \rangle \langle \hat{V} \rangle}(\langle \hat{V}^2 \rangle - \langle \hat{V} \rangle^2)$$

$$\frac{d\langle \hat{H}^2 \rangle}{dt} = \sum_{\alpha=1}^{7} \frac{\langle [\hat{H}^2, \hat{p}_\alpha] \rangle}{i\hbar} = -\frac{\partial p}{\partial \langle \hat{H} \rangle} 2\langle \hat{H} \rangle - 2\frac{\partial^2 p}{\partial \langle \hat{H} \rangle \langle \hat{V} \rangle}(\langle \hat{H}^2 \rangle - \langle \hat{H} \rangle^2)$$

$$\frac{d\langle \hat{H}\hat{V} + \hat{V}\hat{H} \rangle}{dt} = \sum_{\alpha=1}^{7} \frac{\langle [\hat{H}\hat{V} + \hat{V}\hat{H}, \hat{p}_\alpha] \rangle}{i\hbar} = \frac{\partial p}{\partial \langle \hat{H} \rangle} 2\langle \hat{H} \rangle - \frac{\partial p}{\partial \langle \hat{V} \rangle} 2\langle \hat{V} \rangle + \frac{\partial^2 p}{\partial \langle \hat{H} \rangle^2} 4(\langle \hat{H}^2 \rangle - \langle \hat{H} \rangle^2) + \frac{\partial^2 p}{\partial \langle \hat{V} \rangle^2} 2(\langle \hat{V}^2 \rangle - \langle \hat{V} \rangle^2)$$

For the V and H expectation value derivatives, we will consider the third order terms of the Taylor expansion.

$$\frac{d\langle \hat{V} \rangle}{dt} = \sum_{\alpha=1}^{11} \frac{\langle [\hat{V}, \hat{p}_\alpha] \rangle}{i\hbar} = \frac{\partial p}{\partial \langle \hat{H} \rangle} + \sum_{\alpha=8}^{11} \frac{\langle [\hat{V}, \hat{p}_\alpha] \rangle}{i\hbar}$$

$$\frac{d\langle \hat{H} \rangle}{dt} = \sum_{\alpha=1}^{11} \frac{\langle [\hat{H}, \hat{p}_\alpha] \rangle}{i\hbar} = -\frac{\partial p}{\partial \langle \hat{V} \rangle} + \sum_{\alpha=8}^{11} \frac{\langle [\hat{H}, \hat{p}_\alpha] \rangle}{i\hbar}$$

Explicitly, the third order terms of the Taylor expansion are

$$p_8 + p_9 + p_{10} + p_{11} = \frac{1}{6}\frac{\partial^3 p(\langle\hat{H}\rangle, \langle\hat{V}\rangle)}{\partial\langle\hat{H}\rangle^3}(\hat{H} - \langle\hat{H}\rangle)^3 + \frac{1}{6}\frac{\partial^3 p(\langle\hat{H}\rangle, \langle\hat{V}\rangle)}{\partial\langle\hat{V}\rangle^3}(\hat{V} - \langle\hat{V}\rangle)^3$$

$$+\frac{1}{6}\frac{\partial^3 p(\langle\hat{H}\rangle, \langle\hat{V}\rangle)}{\partial\langle\hat{H}\rangle^2\partial\langle\hat{V}\rangle}\Big((\hat{H} - \langle\hat{H}\rangle)^2(\hat{V} - \langle\hat{V}\rangle) + (\hat{H} - \langle\hat{H}\rangle)(\hat{V} - \langle\hat{V}\rangle)(\hat{H} - \langle\hat{H}\rangle) + (\hat{V} - \langle\hat{V}\rangle)(\hat{H} - \langle\hat{H}\rangle)^2\Big)$$

$$+\frac{1}{6}\frac{\partial^3 p(\langle\hat{H}\rangle, \langle\hat{V}\rangle)}{\partial\langle\hat{H}\rangle\partial\langle\hat{V}\rangle^2}\Big((\hat{H} - \langle\hat{H}\rangle)(\hat{V} - \langle\hat{V}\rangle)^2 + (\hat{V} - \langle\hat{V}\rangle)(\hat{H} - \langle\hat{H}\rangle)(\hat{V} - \langle\hat{V}\rangle) + (\hat{V} - \langle\hat{V}\rangle)^2(\hat{H} - \langle\hat{H}\rangle)\Big)$$

After solving, we see that the third order time derivative solutions are as follows

$$\frac{d\langle\hat{V}\rangle}{dt} = \sum_{\alpha=1}^{11}\frac{\langle[\hat{V}, \hat{p}_\alpha]\rangle}{i\hbar} = \frac{\partial p}{\partial\langle\hat{H}\rangle} + \sum_{\alpha=8}^{11}\frac{\langle[\hat{V}, \hat{p}_\alpha]\rangle}{i\hbar}$$

$$= \frac{\partial p}{\partial\langle\hat{H}\rangle} + \frac{1}{6}\Big(\frac{\partial^3 p}{\partial\langle\hat{H}\rangle^3}\Big(3(\langle\hat{H}^2\rangle - \langle\hat{H}\rangle^2)\Big) + \frac{\partial^3 p}{\partial\langle\hat{H}\rangle^2\partial\langle\hat{V}\rangle}\Big(3(\langle\hat{H}\hat{V}\rangle - 2\langle\hat{H}\rangle\langle\hat{V}\rangle + \langle\hat{V}\hat{H}\rangle)\Big) + \frac{\partial^3 p}{\partial\langle\hat{H}\rangle\partial\langle\hat{V}\rangle^2}\Big(3(\langle\hat{V}^2\rangle - \langle\hat{V}\rangle^2)\Big)\Big)$$

$$\frac{d\langle\hat{H}\rangle}{dt} = \sum_{\alpha=1}^{11}\frac{\langle[\hat{H}, \hat{p}_\alpha]\rangle}{i\hbar} = -\frac{\partial p}{\partial\langle\hat{V}\rangle} + \sum_{\alpha=8}^{11}\frac{\langle[\hat{H}, \hat{p}_\alpha]\rangle}{i\hbar} = \frac{\partial p}{\partial\langle\hat{H}\rangle} + \frac{1}{6}\Big(\frac{\partial^3 p}{\partial\langle\hat{H}\rangle^3}\Big(0\Big) + \frac{\partial^3 p}{\partial\langle\hat{V}\rangle^3}\Big(-3(\langle\hat{V}^2\rangle - \langle\hat{V}\rangle^2)\Big)$$

$$+\frac{\partial^3 p}{\partial\langle\hat{H}\rangle^2\partial\langle\hat{V}\rangle}\Big(-3(\langle\hat{H}^2\rangle - \langle\hat{H}\rangle^2)\Big) + \frac{\partial^3 p}{\partial\langle\hat{H}\rangle\partial\langle\hat{V}\rangle^2}\Big(-3(\langle\hat{H}\hat{V}\rangle - 2\langle\hat{H}\rangle\langle\hat{V}\rangle + \langle\hat{V}\hat{H}\rangle)\Big)\Big)$$

Analytical and Numerical Solutions

With the Taylor expansion completed and the time derivatives solved for algebraically, we can substitute our choices for p, namely p_T and p_ϕ, to find dV/dt for each case. After resolving and simplifying the derivatives, we get

$$\frac{dV}{dT} = \frac{3}{4\pi G}H$$

$$\frac{dV}{d\phi} = \frac{3HV}{4\pi G\sqrt{\frac{3H^2}{4\pi G} - \Lambda}}$$

$$+\frac{1}{6}V\frac{27H\Lambda}{(4\pi G)^2(\frac{3H^2}{4\pi G} - \Lambda)^{5/2}}h_2$$

$$+\frac{1}{6}\Big(\frac{3}{4\pi G}\Big)^2\frac{H}{(\frac{3H^2}{4\pi G} - \Lambda)^{3/2}}\left(\frac{\frac{3H^2}{4\pi G} + \Lambda}{\frac{3H^2}{4\pi G} - \Lambda}\right)h_3$$

We can see that dV/dϕ has a much more complicated form than does dV/dT. However, observe that the second term in dV/dϕ approaches zero and the third term is of order $1/H^2$ for small Λ. This is a reasonable approximation due to the fact that the observed cosmological constant is very small. As a result, we exclude the second and third terms from dV/dϕ for our purposes. We will check this approximation numerically.

For the cases of dH/dϕ and dH/dT, we get

$$\frac{dH}{dT} = \frac{p_\phi^2}{V^3} + \frac{2h_1 p_\phi^2}{V^5}$$

$$\frac{dH}{d\phi} = \sqrt{\frac{3H^2}{4\pi G} - \Lambda} - \frac{1}{6}\left(\frac{3}{4\pi G}\right)^2 \frac{H}{(\frac{3H^2}{4\pi G})^{3/2} - \Lambda}\left(\frac{\frac{3H^2}{4\pi G} + \Lambda}{\frac{3H^2}{4\pi G} - \Lambda}\right) h_3$$

Observe that for dH/dT, the second term decays as $1/V^5$. So for large V (an older universe), the second term is much smaller than the first. The second term in dH/dϕ is approximately zero due to the same reasoning as in the case of dV/dϕ.

$$\frac{dV}{dT} = \frac{3}{4\pi G} H, \quad \frac{dH}{dT} = \frac{p_\phi^2}{V^3}$$

$$\frac{d^2V}{dT^2} = \frac{3}{4\pi G}\frac{dH}{dT} \rightarrow \frac{d^2V}{dT^2} = \frac{3}{4\pi G}\frac{p_\phi^2}{V^3}$$

We solve for V and dV/dT.

$$V = \sqrt{\frac{1 + t^2 C_1^2 + 2tC_1^2 C_2 + C_1^2 C_2^2}{C_1}}$$

$$\frac{dV}{dT} = \frac{tC_1^2 + C_1^2 C_2}{\sqrt{C_1}\sqrt{1 + t^2 C_1^2 + 2tC_1^2 C_2 + C_1^2 C_2^2}} = \frac{3}{4\pi G} H$$

After solving $V_0 = V(0)$ and $H_0 = H(0)$, we get

$$C_1 = \frac{1 + V_0^2 H_0^2}{V_0^2}, \quad C_2 = \frac{H_0/V_0}{1 + V_0^2 H_0^2}$$

Now we have to transition to proper time.

$$\frac{dT}{d\tau} = V$$

Integrating dT/dτ gives us

$$T(\tau) = \frac{1}{C_1}\sinh(\sqrt{C_1}\tau) - C_2$$

$$\frac{dV}{dT} = \frac{3H}{4\pi G} \rightarrow \frac{dV}{d\tau} = \frac{3}{4\pi G} HV$$

$$\frac{\dot{V}}{V} = \frac{3}{4\pi G} H \qquad (1)$$

Now we consider the ϕ system.

$$\frac{dV}{d\phi} = \frac{3HV}{4\pi G\sqrt{\frac{3H^2}{4\pi G} - \Lambda}}$$

$$\frac{dH}{d\phi} = -\sqrt{\frac{3H^2}{4\pi G} - \Lambda}$$

$$\frac{d\phi}{d\tau} = \frac{p_\phi}{V}$$

One can still solve for V and H, but the process is much longer. Instead of solving the ϕ equations, one can look at the equations and rewrite $dV/d\phi$ as the expansion rate.

$$\frac{dV}{d\phi} = \frac{3HV}{4\pi G\sqrt{\frac{3H^2}{4\pi G} - \Lambda}}$$

$$\frac{dV}{d\tau} = \frac{3Hp_\phi}{4\pi G\sqrt{\frac{3H^2}{4\pi G} - \Lambda}}$$

$$\frac{\dot{V}}{V} = \frac{3}{4\pi G}H \qquad (2)$$

Since proper time is measured, it should be the same between reference frames. Classically, V/V is indeed the same between our two frames. This can be seen by comparing (1) and (2).

As we bring in the quantum correction terms to the T system, the V equation is unchanged, but the H equation is changed. In the ϕ system, both equations change. Even though (1) and (2) are the same classically, they will be different when quantum correction terms are introduced.

Discussion and Conclusions

As the main result, we have shown that the quantum corrections result in differences between the T and ϕ systems. This result is independent of the approximations made in the Analytical and Numerical Solutions section. If one were interested in complete solutions, one would have to use approximations or full numerics.

After resolving the derivatives from the Taylor expansion, the constant terms h_1, h_2, and h_3 made it impossible to find closed form solutions for our equations. Because of this, we used numerics to find solutions. We used Mathematica's standard procedure for numerical analysis and were able to determine appropriate initial conditions for our systems.

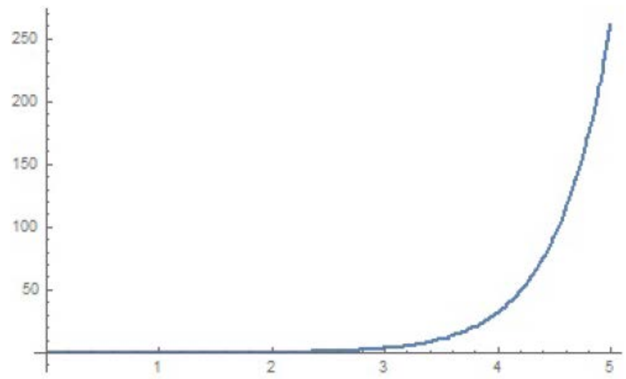

FIG. 1. $(\Delta V)^2$ as a function of τ

Figure 1. Plots the volume fluctuations of the universe with respect to time. The fluctuations grow rapidly, which may not be expected in a classical universe. However, the volume expectation value is increasing as well.

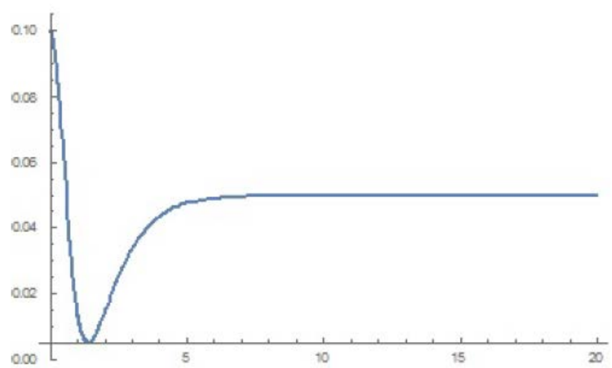

FIG. 2. $(\Delta V)^2/V^2$ as a function of τ

Figure 2. Plots the volume fluctuations of the universe over the volume of the universe with respect to proper time. This agrees with semi-classical behavior because the ratio $(\Delta V)^2/V^2$ is a small constant at large τ. Qualitatively, this agrees with the observed universe.

References

[1] M. Bojowald, Reports on Progress in Physics 78 (2015) 023901

[2] D.F. Styer, American Journal of Physics 58, 742 (1990); doi: 10.1119/1.16396