Some Overpartition k-tuple Congruence Properties

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1 Historical Background and Basic Theory

1.1 What is an ``integer partition''?

The following definition is offered in [1]:

Definition 1.1 *An **integer partition** is a way of splitting a number into integer parts.*

The idea is very simple, as shown in the following example. Consider the integer 5. The question to ask is: how can 5 be written as a combination of positive integers less than or equal to 5? By simple inspection, we yield the following list of answers:

$$5,\ 4+1,\ 3+2,\ 3+1+1,\ 2+2+1,\ 2+1+1+1,\ 1+1+1+1+1$$

Each of the elements of this list is itself a partition, and the numbers in each partition are referred to as **parts** . As one may clearly deduce, the larger the number, the more partitions it has. Also of note is the fact that $2+1+1+1$ is the same as $1+2+1+1$, that is, order is not accounted for. Typically, the parts of a partition are written in nonincreasing order. It is also commonly accepted that the function $p(n)$, where n is a positive integer, is the function which counts the number of partitions of n.

The next question typically asked by one who studies partitions is, ``What different restrictions can I put on two sets of partitions to yield the same number of partitions?" As described in [1], the great mathematician Leonhard Euler asked this question and provided a host of solutions, most commonly referred to as Euler Identities. For example, let's again consider the number 5. Consider first the partitions of 5 that contain only distinct parts:

$$5,\ 4+1,\ 3+2$$

Now, consider partitions of 5 containing only odd parts:

$$5,\ 3+1+1,\ 1+1+1+1+1$$

As is easily seen, both of the lists contain the same number of partitions. Indeed, one can show that for any positive number, the number of partitions into odd parts is exactly the number of partitions into distinct parts. In fact, the following stronger statement is true, where $p(n)$ is the number of partitions of n:

Theorem 1.2 (Euler Pairs)

$$p(n \text{ with parts in N}) = p(n \text{ with distinct parts in M}) \quad for \;\; n \geq 1,$$

where N is any set of integers such that no element of N is a power of two times an element of N, and M is the set containing all elements of N together with all their multiples of powers of two.

1.2 Congruences

Definition 2.1 *Consider two integers m and n. If n has remainder r when divided by m, then n is said to be* ***congruent*** *to r modulo m, denoted by*

$$n \equiv r \pmod{m}.$$

In the above definition, if n is divisible by m, then this means $r = 0$. Thus $n \equiv 0 (\text{mod } m)$. For further understanding, refer to the following example.

Example 1 *Let* $m = 4$. *Let* $a = 1$, $b = 6$, $c = -1$, $d = 16$, $e = 26$, *and* $f = -15$. *Then*

$$a \equiv 1 \pmod{4}, \qquad d \equiv 0 \pmod{4},$$
$$b \equiv 2 \pmod{4}, \qquad e \equiv 2 \pmod{4},$$
$$c \equiv 3 \pmod{4}, \qquad f \equiv 1 \pmod{4}.$$

1.3 Generating Functions

As discussed in [1], the entire idea of generating functions for integer partitions lies solely on the following fact:

$$q^a \times q^b = q^{a+b}$$

To use this fact in finding all integer partitions with one even part and one odd part, each of which is less than 5, consider the following:

$$(q^2+q^4)(q^1+q^3) = q^{2+1}+q^{2+3}+q^{4+1}+q^{4+3} \tag{1.3.1}$$
$$= q^3+q^5+q^5+q^7 \tag{1.3.2}$$
$$= q^3+2q^5+q^7. \tag{1.3.3}$$

Upon further inspection, it is obvious that (1.3.1) lists in each exponent all of the partitions satisfying the aforementioned condition. This can be generalized. Let $S = \{n_1, n_2, n_3\}$, with each n_i a positive integer. Then consider

$$(1+q^{n_1})(1+q^{n_2})(1+q^{n_3}) = 1+q^{n_1}+q^{n_2}+q^{n_3}+q^{n_1+n_2}+q^{n_1+n_3}+q^{n_2+n_3}+q^{n_1+n_2+n_3}. \quad (1.3.4)$$

This expression displays all partitions using distinct members of the set S.

Example 2 *Let* $S = \{1,2,3\}$. *Then the polynomial from (1.3.4) becomes:*

$$1+q+q^2+2q^3+q^4+q^5+q^6$$

With that in mind, the following definition is now available.

Definition 3.1 *A polynomial or power series whose coefficients represent the number of partitions of its exponents is called a* ***generating function*** *. This function also allows for certain restrictions to be placed upon the parts.*

In Example 2, the generating function is counting the number of partitions into distinct elements from the set S. This leads to the following fact for a set of positive integers $S = \{n_1, n_2, \dots, n_r\}$:

$$\sum_{n\geq 0} p(n \text{ with distinct parts in } S)q^n = \prod_{i=1}^{r}(1+q^{n_i}) = \prod_{n\in S}(1+q^n).$$

This can also be used if we want to allow parts to only repeat a certain number of times. For instance, suppose we want each part to appear up to 3 times in any partition, and that we only want to use parts from $S = \{n_1, n_2\}$. Then,

$$(1+q^{n_1}+q^{n_1+n_1}+q^{n_1+n_1+n_1})(1+q^{n_2}+q^{n_2+n_2}+q^{n_2+n_2+n_2})$$

$$\begin{aligned}= 1 &+q^{n_2}+q^{n_2+n_2}+q^{n_2+n_2+n_2}+q^{n_1}+q^{n_1+n_2}+q^{n_1+n_2+n_2}\\ &+q^{n_1+n_2+n_2+n_2}+q^{n_1+n_1}+q^{n_1+n_1+n_2}+q^{n_1+n_1+n_2+n_2}\\ &+q^{n_1+n_1+n_2+n_2+n_2}+q^{n_1+n_1+n_1}+q^{n_1+n_1+n_1+n_2}+q^{n_1+n_1+n_1+n_2+n_2}\\ &+q^{n_1+n_1+n_1+n_2+n_2+n_2}\end{aligned}$$

$$= \sum_{n\geq 0} p(n \text{ with parts in } \{n_1, n_2\}, \text{no part repeated more than 3 times})q^n.$$

If $S = \{n_1, \dots, n_r\}$, then

$$\begin{aligned}\sum_{n\geq 0} p(n \text{ with parts in } S, \text{none repeated more than } d \text{ times})q^n\\ = \prod_{i=1}^{r}\left(1+q^{n_i}+q^{n_i+n_i}+\cdots+q^{\overbrace{n_i+\cdots+n_i}^{d\,\text{times}}}\right)\\ = \prod_{i=1}^{r}\left(1+q^{n_i}+q^{2n_i}+\cdots+q^{dn_i}\right)\\ = \prod_{i=1}^{r}\frac{(1-q^{(d+1)n_i})}{(1-q^{n_i})} = \prod_{n\in S}\frac{1-q^{(d+1)n}}{1-q^n}.\end{aligned}$$

The end of the previous list of equalities is true due to the following known formula for the finite geometric series:

$$\sum_{j=0}^{N} x^j = \frac{1-x^{N+1}}{1-x}$$

By letting $d \to \infty$, we can see that the generating function is still meaningful for letting parts appear an arbitrary number of times. We now must require $|q| < 1$, which is not a problem since we do not substitute a value for q. So, for $|q| < 1$,

$$\sum_{n\geq 0} p(n \text{ with parts in } S)q^n = \prod_{i=1}^{\infty} \left(1 + q^{n_i} + q^{2n_i} + \cdots\right)$$
$$= \prod_{i=1}^{\infty} \frac{1}{1-q^{n_i}} = \prod_{n\in S} \frac{1}{1-q^{n}}.$$

Here, the formula for an infinite geometric series was used:

$$\sum_{j=0}^{\infty} x^j = \frac{1}{1-x}, \quad |x| < 1$$

Thus, we have a representation for the generating functions of the number of partitions for many more cases than we did originally. These generating functions are instrumental for successfully proving results.

2 Definitions, Notation and Terminology

2.1 What is an ``overpartition''?

In the study of partitions there are always new constructions appearing. Consider $n = 5$. We already know its partitions:

$$5,\ 4+1,\ 3+2,\ 3+1+1,\ 2+2+1,\ 2+1+1+1,\ 1+1+1+1+1$$

Our new objective is to expand these results into some sort of new idea, then see if our new list has any interesting properties. And, if our new construction is done well enough, our ultimate goal is to relate it back to unrestricted integer partitions. So, let's consider a new list, with each element abiding by these rules:

- It is a partition of n.
- The first occurence of any part may be $\overline{\text{overlined}}$.

So, how does this change our list for $n = 5$? Our new list is as follows:

$$5,\ 4+1,\ 3+2,\ 3+1+1,\ 2+2+1,\ 2+1+1+1,$$
$$1+1+1+1+1,\ \overline{5},\ \overline{4}+\overline{1},\ \overline{4}+1,\ 4+\overline{1},\ \overline{3}+\overline{2},$$
$$\overline{3}+2,\ 3+\overline{2},\ \overline{3}+\overline{1}+1,\ \overline{3}+1+1,\ 3+\overline{1}+1,$$
$$\overline{2}+2+\overline{1},\ \overline{2}+2+1,\ 2+2+\overline{1},\ \overline{2}+\overline{1}+1+1,$$
$$2+1+1+1,\ 2+1+1+1,\ 1+1+1+1+1$$

This example motivates the following definition, as stated in [6].

Definition 1.1 *An* ***overpartition*** *of n is a non-increasing sequence of natural* *numbers whose sum is n in which the first occurrence of a number may be overlined.*

Throughout, the number of overpartitions of a number n will be denoted by $\overline{p}(n)$, with $\overline{p}(0) = 1$. Also from [6] we have the following generating function for $\overline{p}(n)$:

$$\sum_{n=0}^{\infty} \overline{p}(n) q^n = \prod_{n=1}^{\infty} \frac{1+q^n}{1-q^n} = 1 + 2q + 4q^2 + 8q^3 + 14q^4 + \cdots \qquad (5)$$

2.2 Overpartition Pairs and k-tuples

Overpartitions have been studied quite a bit recently, such as in [4], [5], [7], [9], and [12]. As an expansion of overpartitions, the following new construction appeared in [10].

Definition 2.1 *An* ***overpartition pair*** (λ, μ) *of n is a pair of overpartitions where the sum of all listed parts is n.*

Example 3 *The overpartition pairs for* $n = 2$ *are:*

$(2\ ;\ \emptyset)$, $(\overline{2}\ ;\ \emptyset)$, $(1+1\ ;\ \emptyset)$, $(\overline{1}+1\ ;\ \emptyset)$,
$(1\ ;\ 1)$, $(1\ ;\ 1)$, $(1\ ;\ 1)$, $(1\ ;\ 1)$, $(\emptyset\ ;\ 2)$,
$(\emptyset\ ;\ 2)$, $(\emptyset\ ;\ 1+1)$, $(\emptyset\ ;\ \overline{1}+1)$.

The number of overpartition pairs of n were denoted in [10] by $\overline{pp}(n)$. Also in [10], it was discovered that the generating function for $\overline{pp}(n)$ is

$$\sum_{n=0}^{\infty} \overline{pp}(n) q^n = \prod_{n=1}^{\infty} \left(\frac{1+q^n}{1-q^n}\right)^2 = 1 + 4q + 12q^2 + 32q^3 + 76q^4 + \cdots.$$

This generating function was derived from the generating function for overpartitions. Noting how similar this generating function looks compared to the generating function for $\overline{p}(n)$, one can now expand on this idea easily.

Definition 2.2 *An* ***overpartition k-tuple*** *of n is a k-tuple of overpartitions* $(\lambda_1, \dots, \lambda_k)$ *wherein all listed parts sum to n.*

We will denote the number of overpartition k-tuples by $\overline{p}_k(n)$, which subsequently means $\overline{pp}(n) = \overline{p}_2(n)$. Based on the generating function for overpartition pairs, it is clear that the generating function for overpartition k-tuples is

$$\sum_{n \geq 0} \overline{p}_k(n) q^n = \prod_{n=1}^{\infty} \left(\frac{1+q^n}{1-q^n}\right)^k.$$

3 Known Congruences

3.1 Ramanujan

The great mathematician Srinivasa Ramanujan (1887-1920) discovered and proved countless theorems in many fields, including number theory. Directly pertinent to this study, and providing most of the inspiration, are the following congruences that Ramanujan proved about partitions.

Theorem 1.1 *For all* $n \geq 0$,

$$p(5n+4) \equiv 0 \pmod{5},$$
$$p(7n+5) \equiv 0 \pmod{7}, \text{and}$$
$$p(11n+6) \equiv 0 \pmod{11}.$$

For example, here are tables showing $p(n)$ for some values of $5n+4$, $7n+5$, and $11n+6$.

n	$5n+4$	$p(5n+4)$
0	4	5
1	9	30
2	14	135
3	19	490
4	24	1575

n	$7n+5$	$p(7n+5)$
0	5	7
1	12	77
2	19	490
3	26	2436
4	33	10143

n	$11n+6$	$p(11n+6)$
0	6	11
1	17	297
2	28	3718
3	39	31185
4	50	204226

3.2 Overpartition and Overpartition Pair Congruences

Similar to the above, there are many interesting congruences related to $\overline{p}(n)$ and $\overline{p}_2(n)$. Some are straightforward, and others take a bit of work to prove. The following is a table of the first few values of the overpartition function.

n	$\overline{p}(n)$
1	2
2	4
3	8
4	14
5	24
6	40
7	64
8	100
9	154
10	232

Clearly, it appears that all of the values for $\overline{p}(n)$ are even. In fact, the next theorem proves just that.

Theorem 2.1 *For all* $n \geq 1$, $\overline{p}(n) \equiv 0 \pmod{2}$.

Proof.

$$\sum_{n=0}^{\infty} \overline{p}(n) q^n = \prod_{i=1}^{\infty} \frac{1+q^i}{1-q^i} \equiv \prod_{i=1}^{\infty} \frac{1+q^i-2q^i}{1-q^i} \pmod{2}$$

$$= \prod_{i=1}^{\infty} \frac{1-q^i}{1-q^i}$$

$$= 1.$$

$$\text{Therefore, } \sum_{n=1}^{\infty} \overline{p}(n) q^n \equiv 1 \pmod{2}$$

$$\equiv 1q^0 + 0q^1 + 0q^2 + 0q^3 + \cdots \pmod{2}.$$

Since the coefficients of the terms on the right correspond to the same ones on the left, we see that for $n \geq 1$, $\overline{p}(n) \equiv 0 \pmod{2}$.

The next theorem follows from results in [11].

Theorem 2.2 *For all* $n > 0$,

$$\overline{p}(n) \equiv \begin{cases} 2 \pmod{4} & \text{if } n \text{ is a square;} \\ 0 \pmod{4} & \text{otherwise.} \end{cases}$$

Here's a table of the first twenty values for the overpartition function. It is easy to see that $p(n) \equiv 0 (\mathrm{mod}\ 4)$ for all n that are not square.

n	$\overline{p}(n)$
1	2
2	4
3	8
4	14
5	24

n	$\overline{p}(n)$
6	40
7	64
8	100
9	154
10	232

n	$p(n)$
11	344
12	504
13	728
14	1040
15	1472

n	$p(n)$
16	2062
17	2864
18	3948
19	5400
20	7336

The next two theorems, proved in [8], are a little more involved.

Theorem 2.3 *Let* $\overline{p_o}(n)$ *be the number of overpartitions of n into odd parts. Then*

$$\overline{p_o}(n) \equiv \begin{cases} 2 \pmod{4} & \text{if } n \text{ is a square or if } n \text{ is twice a square;} \\ 0 \pmod{4} & \text{otherwise.} \end{cases}$$

Theorem 2.4 *Let* $\overline{p_o}(n)$ *be the number of overpartitions of n into odd parts. Then, for all* $n \geq 0$ *and for all* $\alpha \geq 0$,

$$\overline{p_o}\left(9^{\alpha}(9n+6)\right) \equiv 0 \pmod{12} \quad \text{and}$$

$$\overline{p_o}\left(9^{\alpha}(27n+9)\right) \equiv 0 \pmod{6}.$$

Finally, we have a theorem in the spirit of Ramanujan about overpartition pairs, as proven in [2].

Theorem 2.5 *For all natural numbers n,*

$$\overline{p}_2(3n+2) \equiv 0 \pmod 3.$$

What follows is a table of the first 15 values for the overpartition pairs function. In checking the function for values of $3n+2$, one can see that they are all divisible by 3.

n	$\overline{p}_2(n)$
1	4
2	12
3	32
4	76
5	168

n	$\overline{p}_2(n)$
6	352
7	704
8	1356
9	2532
10	4600

n	$\overline{p}_2(n)$
11	8160
12	14176
13	24168
14	40512
15	66880

4 New Results

The major goal of this research is to see if previous results about overpartitions and their pairs could be extended to similar results for overpartition k-tuples. The forthcoming results accomplished this goal and more. In fact, a few of the aforementioned theorems can be more easily proven now, as they are simply corollaries of the new results.

Theorem 4.1 *For all n>0 and all nonnegative integers m, we have*

$$\overline{p}_{2^m}(n) \equiv 0 \pmod{2^{m+1}}.$$

Proof.

$$\sum_{n=0}^{\infty} \overline{p}_{2^m}(n)q^n = \prod_{i=1}^{\infty} \left[\frac{1+q^i}{1-q^i}\right]^{2^m}$$

$$\equiv \prod_{i=1}^{\infty} \left[\frac{1+q^i+2^{m+1}q^i}{1-q^i}\right]^{2^m} \pmod{2^{m+1}}$$

$$= \prod_{i=1}^{\infty} \left[1+\frac{(2^{m+1}+2)q^i}{1-q^i}\right]^{2^m}$$

$$= \prod_{i=1}^{\infty} \left[1+2\left(\frac{(2^m+1)q^i}{1-q^i}\right)\right]^{2^m}$$

$$= \prod_{i=1}^{\infty} \left[\sum_{n=0}^{2^m} \binom{2^m}{n} 2^n \left(\frac{(2^m+1)q^i}{1-q^i}\right)^n\right] \text{ by [3, Theorem 5.2.1]}$$

$$= \prod_{i=1}^{\infty} \left[1+2^{m+1}\left(\frac{(2^m+1)q^i}{1-q^i}\right)+2^{m+1}(2^m-1)\left(\frac{(2^m+1)q^i}{1-q^i}\right)^2\right.$$

$$\left.+\cdots+2^{(m+2^m-1)}\left(\frac{(2^m+1)q^i}{1-q^i}\right)^{2^m-1}+2^{2^m}\left(\frac{(2^m+1)q^i}{1-q^i}\right)^{2^m}\right]$$

$$\equiv 1 \pmod{2^{m+1}}$$

Thus, the congruence shown is

$$\sum_{n=0}^{\infty} \overline{p}_{2^m}(n)q^n \equiv 1 \pmod{2^{m+1}}$$

$$\equiv 1 + 0q + 0q^2 + 0q^3 + \cdots \quad (\text{mod } 2^{m+1}).$$

Since coefficients on either side correspond, and $\overline{p}_{2^m}(0) = 1$ by definition, the preceding implies

$$\overline{p}_{2^m}(n) \equiv 0 (\text{mod } 2^{m+1})$$

for all $n > 0$ and for all nonnegative integers m.

With this theorem in hand, a broader theorem is easily proved.

Theorem 4.2 *Let $\overline{p}_k(n)$ count the number of overpartition k-tuples of n. Let k=$(2^m)r$, where m is a nonnegative integer and r is odd. Then, for all positive integers n,*

$$\overline{p}_k(n) \equiv 0 (\text{mod } 2^{m+1}).$$

Proof.

$$\sum_{n=0}^{\infty} \overline{p}_k(n) q^n = \prod_{i=1}^{\infty} \left[\frac{1+q^i}{1-q^i}\right]^k = \prod_{i=1}^{\infty} \left[\frac{1+q^i}{1-q^i}\right]^{(2^m)r}$$

$$= \prod_{i=1}^{\infty} \left(\left[\frac{1+q^i}{1-q^i}\right]^{2^m}\right)^r$$

$$= \left(\prod_{i=1}^{\infty} \left[\frac{1+q^i}{1-q^i}\right]^{2^m}\right)^r$$

$$\equiv 1^r \quad (\text{mod } 2^{m+1}) \text{ using Theorem 4.1}$$

$$\equiv 1 \quad (\text{mod } 2^{m+1}).$$

Just as in Theorem 4.1, the result follows.

After acquiring Theorems 4.1 and 4.2, it seemed something may also occur when k and the modulus are simply prime. This is indeed the case as the next theorem shows.

Theorem 4.3 *For all $k > 0$ and all n such that $n \not\equiv 0 (\text{mod } r^k)$, where r is an odd prime, we have*

$$\overline{p}_{r^k}(n) \equiv 0 \quad (\text{mod } r).$$

Proof.

$$\sum_{n=0}^{\infty} \overline{p}_{r^k}(n) q^n = \prod_{i=1}^{\infty} \left(\frac{1+q^i}{1-q^i}\right)^{r^k} = \prod_{i=1}^{\infty} \frac{(1+q^i)^{r^k}}{(1-q^i)^{r^k}}$$

$$\equiv \prod_{i=1}^{\infty} \frac{1+q^{ir^k}}{1-q^{ir^k}} (\text{mod } r).$$

Upon expanding this last product as a power series, the only surviving (non-zero) exponents remaining are all multiples of r^k. Thus, $\overline{p}_{r^k}(n) \equiv 0 (\text{mod } r)$ for all positive integers k and for all $n \not\equiv 0 (\text{mod } r^k)$.

5 The Road Ahead

The new results from the last section are only the beginning of the discovery of many results for overpartition k-tuples. From here, the goal is to find other congruences and identities for different overpartition k-tuples. Some possibilities include:

Conjecture 5.1 *For all* $n > 0$*,*

$$p_{q-1}(qn+r) \equiv 0 \pmod{q}$$

where *q* is prime and *r* is a quadratic nonresidue mod *q*.

Theorem 2.5 is a direct corollary of the preceding conjecture.

Conjecture 5.2 *For all integers* $m > 0$*, we have*

$$\overline{p}_{2^m}(n) \equiv \begin{cases} 2^{m+1} & \pmod{2^{m+2}} \quad \text{if } n \text{ is a square or twice a square,} \\ 0 & \pmod{2^{m+2}} \quad \text{otherwise.} \end{cases}$$

Conjecture 5.3 *Let* $\overline{p}_k(n)$ *count the number of overpartition k-tuples of n. Let k=*$(2^m)r$*,* $m > 0$ *and r is odd. Then,*

$$\overline{p}_k(n) \equiv \begin{cases} 2^{m+1} & \pmod{2^{m+2}} \quad \text{if } n \text{ is a square or twice a square,} \\ 0 & \pmod{2^{m+2}} \quad \text{otherwise.} \end{cases}$$

Conjectures 5.2 and 5.3 simply involve raising the power of two in the modulus. They are also inspired by [11]. From here, the intention is to ultimately write a paper co-authored with Dr. Sellers and fellow Penn State student Derrick Keister and publish it in a peer-reviewed journal. It will be submitted by October 2008.

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